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1981 J. Phys. A: Math. Gen. 14 L285

(http://iopscience.iop.org/0305-4470/14/8/006)

J. Phys. A: Math. Gen. 14 (1981) L285–L290. Printed in Great Britain

LETTER TO THE EDITOR

Percolation properties of a three-dimensional random resistor-diode network

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Received 7 May 1981

Abstract. We study percolation in a random resistor-diode network on the simple cubic lattice. Here the occupied bonds joining nearest-neighbour sites may be either resistor-like, transmitting information or connectivity in either direction along a bond, or *diode*-like, transmitting in only one direction. We consider a model in which there exist both positive diodes which transmit in the $+x$, $+y$ or $+z$ directions, and negative diodes which transmit oppositely. We apply position-space renormalisation group methods to map out the phase diagram, and calculate the exponents associated with the phase transitions of this system. We find three novel types of transitions due to the presence of the diodes. With only one species of diodes and vacancies (unoccupied bonds) present, a directed threshold occurs at a critical concentration of diodes. Here an infinite cluster forms which transmits only in the direction of the average diode polarisation. A reverse threshold occurs when resistors and only one species of diodes exist. At this point the resistors mediate information flow in the infinite cluster *opposite* to the diode polarisation. Finally, with all bond elements present, a mixed threshold occurs. At this point, an isotropic infinite cluster exists, but of a qualitatively different character from that occurring at the usual bond percolation threshold. The mixed and isotropic transitions are higher-order critical points where diode, resistor and vacancy phases become simultaneously critical.

The percolation problem has been extensively investigated, partly because it is an extremely simple system exhibiting the intriguing complexities of second-order phase transitions, and also becuse of the many realisations of percolation phenomena in nature. (See e.g. Stauffer (1979), Essam (1980) for recent reviews.) Recently, attention has focused on developing more general percolation models which are a challenge on a fundamental theoretical level, as well as finding applications for percolation in many diverse fields. Many such generalisations are contained implicitly in the early work of Broadbent and Hammersley (1957). They proposed a percolation process in which neighbouring lattice sites may be joined by *two* randomly occupied directed bonds, one 'transmitting' connectivity or information in one direction, and the other transmitting in the reverse direction. In this sense, the directed bonds act as *diodes*, thus breaking the isotropic symmetry of the usual bond percolation problem.

One special case of this Broadbent–Hammersley percolation process is directed bond percolation. For example, on the square lattice, the occupied bonds may transmit only upward or to the right. Above the percolation threshold, the infinite cluster can be traversed from the lower left to the upper right, but not in the reverse direction. This

† Supported in part by grants from the ARO and AFOSR.

model has several interesting features not found in isotropic bond percolation. These include different critical behaviour (Blease 1977a, b, c, Kertész and Vicsek 1980), and an anisotropic structure for the infinite cluster (Obukhov 1980). Because of the latter result, the decay of correlations is characterised by two length scales, one parallel and one perpendicular to the directed axis (Dhar and Barma 1981, Kinzel and Yeomans 1981). Directed percolation has been mapped into a Reggeon field theory (Cardy and Sugar 1980), and the latter model can be related to Markov processes with absorption, branching and recombination (Grassberger and Sundermeyer 1978, Grassberger and de la Torre 1979), which are of relevance for describing many chemical and biological processes (Schlögl 1972).

Very recently, there has been an interest in generalising directed bond percolation to a situation where the orientation of the diodes is random. In this connection, Reynolds (1981) considered a model in which a lattice was randomly occupied by either positive diodes (transmitting upward or to the right), or negative diodes (transmitting in the opposite direction). In addition, each of these diodes could 'break down' with a random probability and transmit in both directions. Redner (1981) treated a similar problem of a random resistor-diode network. In this model, each bond may be independently occupied by positive or negative diodes, resistors and vacancies (empty bonds). In both of these systems, the possibility of continuously varying the diode polarisation leads to a rich phase diagram and novel types of phase transitions. These include thresholds for the formation of an infinite cluster which transmits information unidirectionally, either parallel or antiparallel to the diode polarisation. In addition, there exist isotropic thresholds which exhibit higher-order critical behaviour where resistor, vacancy and two diode phases become simultaneously critical.

Because of the richness of this model, we have extended the study of the physical properties of the random resistor-diode network to the simple cubic lattice. In this Letter, we consider a model system containing positive diodes which transmit in the $+x$, $+y$ or $+z$ directions, and negative diodes which transmit in the opposite direction. Resistors transmit in either direction, and vacancies are non-transmitting (see figure 1(a)). These elements occur with random probabilities d_+, d_-, r and v respectively. In this model, the diode polarisation points along the (1, 1, 1) axis, and can take any value between -1 and $+1$. When the polarisation per occupied bond equals ± 1, we recover directed percolation. On the other hand, when the polarisation equals zero, we have a

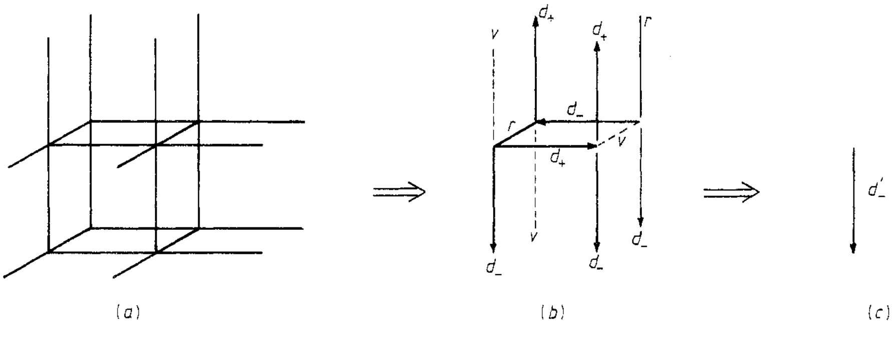

Figure 1. A 2^3 bond cell on the simple cubic lattice. From this we require only the twelve bonds shown in (b) to calculate the probability of traversing the cell. In (b), we represent diodes, resistors and vacancies by arrows, full lines and broken lines respectively. Under rescaling, the configuration in (b) maps to the state shown in (c).

random mixture of resistors, vacancies and diodes which have no net orientational order or polarisation.

We have applied the position-space renormalisation group (PSRG) to study the properties of this random resistor-diode network. Our procedure is based on the simplest approximation of rescaling a 2^3 cell of bonds to a single bond, as shown in figure 1(*b*) (Reynolds *et al* 1977). Configurations which traverse from one edge of the cell to the other, and vice versa, renormalise to a resistor. Configurations which traverse in only one direction renormalise to a diode oriented in the direction of traversing. Finally, non-traversing configurations renormalise to a vacancy. The probabilities for each of these four cases gives the recursion relations for r', $d'_{\pm}$ and v' respectively. Since there are 4^{12} states of the cell, it is not feasible to calculate the recursion relations by hand. Accordingly, we have written a computer program which generates each of these configurations and tests for the existence of both a path traversing from the top to the bottom of the cell, and a path traversing in the opposite direction. From this we assign a given cell configuration and its probability to a particular renormalised state.

The recursion relations thus obtained give the phase diagram shown in figure 2 in the probability space spanned by d_+, d_-, r and v. The probabilities must sum to unity, hence the phase space can be represented by a tetrahedron. For any point inside the tetrahedron, the relative amount of the four bond species is determined by the perpendicular distances between the point and the four faces of the tetrahedron. The distance to any face gives the relative amount of the species labelled at the corner opposite the face (see figure 2). There exist two intersecting surfaces of second-order transitions which divide the diagram into four phases. These are the 'vacancy' phase, in which only finite clusters occur; an isotropic 'resistor' phase in which information flows isotropically within the infinite cluster; and two unidirectional 'diode' phases, in which information flows only along the direction of diode polarisation. Most of the volume of the phase diagram is in the resistor phase. This is in striking contrast to the situation on

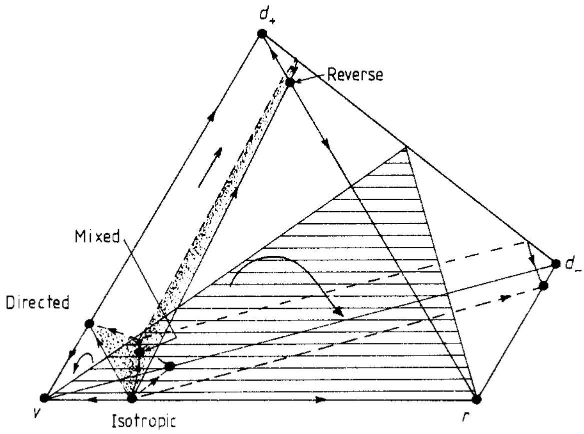

Figure 2. Phase diagram of the random resistor-diode network in the probability space spanned by d_+, d_-, r and v. The fixed points are shown as full circles, and the arrows indicate the direction of flow under renormalisation. The two shaded surfaces form a wedge-shaped structure, which together with an identical wedge on the opposite side of the symmetry plane defined by $d_+ = d_-$, divides the space into four distinct phase regions. The intersection of the two wedges defines a curve which is a line of higher-order critical points where the four phases meet.

the square lattice, where by duality the vacancy and resistor phases occupy equal volumes (Redner 1981). The two second-order surfaces intersect along a curve in the symmetry ($d_+ = d_-$) plane of the phase diagram. This curve is a line of higher-order critical points where the four phases become simultaneously identical. From the recursion relations, we find ten fixed points. Four of them are trivial, and they are associated with a lattice completely filled with only one bond species. Of the remaining six fixed points, there are two directed fixed points which signal the onset of an infinite cluster transmitting information along the direction of polarisation. These are the non-trivial fixed points of the diode-vacancy problem. They occur at d_+^* (or d_-^*) = 0.2392, $v^* = 1 - d_\pm^*$ and $r^* = 0$. Our value of $d_\pm^*$ should be compared with the estimate $d_c = 0.383 \pm 0.003$ found by low-density series (Blease 1977a). To find the exponents, we calculate the linearised transformation matrix $T_{\alpha\beta} \equiv \partial\alpha'/\partial\beta$ ($\alpha, \beta = r, d_+$ or d_-). At the d_+ directed fixed point, $T_{\alpha\beta}$ has the value (to four significant figures)

$$\begin{pmatrix} 0 & 0 & 0 \\ 1.989 & 1.859 & 0.1299 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

Thus there is one relevant thermal eigenvalue, 1.859, associated with the eigenvector in the d_+ direction. From this, we obtain a longitudinal correlation length exponent, $\nu_\parallel = \ln 2/\ln 1.989 = 1.118$, describing the divergence of the length along the diode polarisation. The two irrelevant eigenvalues are zero, indicating that the flow into the fixed point is asymptotically within the plane perpendicular to the d_+ direction, and is isotropic due to the eigenvalue degeneracy.

There also exist two reverse fixed points which signal the onset of an infinite cluster that transmits information opposite to the diode polarisation. These points occur at $d_\pm = 0.8699$, $r^* = 1 - d_\pm$ and $v^* = 0$. At the d_+ reverse fixed point (positive diodes and resistors), $T_{\alpha\beta}$ equals

$$\begin{pmatrix} 1.839 & 0.0380 & 1.802 \\ -1.839 & -0.0380 & -1.802 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

Again there is one relevant eigenvalue, 1.802, associated with the eigenvector $(1, -1, 0)$. This leads to a correlation length exponent of $\nu_r = 1.177$ describing the divergence of the mean length of paths transmitting opposite to the diode polarisation. Also, the situation with the irrelevant eigenvalues is exactly the same as at the directed fixed point.

The isotropic bond percolation threshold occurs at $r^* = 0.2085$, $v^* = 1 - r^*$ and $d_\pm^* = 0$. This value of r^*, also found by previous PSRG studies (Bernasconi 1978, Kirkpatrick 1979), is a good approximation to a series estimate of $r_c = 0.247 \pm 0.003$ (Sykes *et al* 1976). This threshold is a higher-order critical point where the vacancy, resistor and two diode phases become simultaneously critical. At this point the matrix $T_{\alpha\beta}$ is

$$\begin{pmatrix} 1.958 & 0 & 0 \\ 0 & 1.839 & 0.1191 \\ 0 & 0.1191 & 1.839 \end{pmatrix}, \tag{3}$$

leading to three relevant eigenvalues. The eigenvalue 1.958 is doubly degenerate, and is associated with the eigenvectors $(1, 0, 0)$ and $(0, 1, 1)$, which lie in the d_+, d_- symmetry plane. Thus if we approach the fixed point along any direction in this plane, we find the correlation length diverging isotropically with an exponent of 1.032. The value of ν is a good approximation to the series expansion estimates of 0.82 ± 0.02 (see e.g. Essam 1980). The unique eigenvalue 1.720 is associated with an eigenvector $(0, 1, -1)$. This vector points outside the physical parameter space, and it is not clear how to interpret the exponents associated with this direction.

The most interesting feature of this phase diagram is the fixed point at $d_+^* = d_-^* = 0.1658$, $r^* = v^* = 0.0446$. This 'mixed' threshold signals the onset of an isotropic infinite cluster whose connectivity requires both diodes and resistors. Even though the infinite cluster is isotropic, there exist two independent associated correlation lengths. One is an isotropic length scale as the fixed point is approached in the 'r' direction. The second length scale is associated with connected paths transmitting opposite to the diode polarisation as the fixed point is approached in the 'd' direction. As the polarisation approaches 0, this length diverges differently from the isotropic length. From the linearised recursion relations

$$\begin{pmatrix} 1.064 & 0.3541 & 0.3541 \\ 0.9077 & 1.497 & -0.2335 \\ 0.9077 & -0.2335 & 1.497 \end{pmatrix} \tag{4}$$

we find two relevant eigenvalues, 1.972 and 1.731, associated with the eigenvectors $(0.7801, -1, -1)$ and $(0, 1, -1)$, and one irrelevant eigenvalue of 0.3557 associated with $(1, -1, -1)$. The relevant eigenvalues yield an exponent of 1.021 describing the divergence of the isotropic length, and an exponent of 1.264 describing the divergence of the unidirectional length. The mixed fixed point is a domain of attraction for a curve lying within the symmetry plane along which all four phases of the system are simultaneously critical. Thus the mixed fixed point is a higher-order critical point in the phase diagram.

In summary, we have studied a generalised percolation problem where lattice bonds can be occupied by positive or negative diodes, or by resistors. This system exhibits novel critical behaviour, and we have applied the position-space renormalisation group to calculate the exponents associated with its phase transitions. We find that there exist directed and reverse thresholds where an infinite cluster forms that transmits connectivity either parallel or antiparallel to the diode polarisation respectively. In addition, a new mixed transition occurs. Here an isotropic infinite cluster forms whose connectivity requires both resistors and randomly oriented diodes. This point, along with the usual bond percolation threshold, are higher-order critical points where the vacancy, resistor and two diode phases become simultaneously critical. At the mixed fixed point, we have found different exponents associated with approaching the fixed point in two independent directions. Along one, the diode polarisation is varied at fixed resistor concentration and a unidirectional correlation length diverges , while along the other, the resistor concentration is varied at fixed polarisation and an isotropic length scale diverges.

References

Bernasconi J 1978 *Phys. Rev.* B **18** 2185
Blease J 1977a *J. Phys. C: Solid State Phys.* **10** 917
—— 1977b *J. Phys. C: Solid State Phys.* **10** 925
—— 1977c *J. Phys. C: Solid State Phys.* **10** 3461

Broadbent S R and Hammersley J M 1957 *Proc. Camb. Phil. Soc.* **53** 629
Cardy J L and Sugar R L 1980 *J. Phys. A: Math. Gen.* **13** L423
Dhar D and Barma M 1981 *J. Phys. C: Solid State Phys.* **14** L5
Essam J W 1980 *Rep. Prog. Phys.* **43** 833
Grassberger P and Sundermeyer K 1978 *Phys. Lett.* **77B** 220
Grassberger P and de la Torre A 1979 *Ann. Phys.* **122** 373
Kertész J and Vicsek T 1980 *J. Phys. C: Solid State Phys.* **13** L343
Kinzel W and Yeomans J M 1981 *Preprint*
Kirkpatrick S 1979 in *Proc. Les Houches Summer School of Ill Condensed Matter* (Amsterdam: North-Holland)
Obukhov S P 1980 *Physica* **101A** 145
Redner S 1981 *Preprint*
Reynolds P J 1981 *Preprint*
Reynolds P J, Klein W and Stanley H E 1977 *J. Phys. C: Solid State Phys.* **10** L167
Schlögl F 1972 *Z. Phys.* **253** 14
Stauffer D 1979 *Phys. Rep.* **54** 1
Sykes M F, Gaunt D S and Glen M 1976 *J. Phys. A: Math. Gen.* **9** 1705